Generators and closed classes of groups

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January 15th, 2017

Abstract

We show that in the category of groups, every singly-generated class which is closed under isomorphisms, direct limits and extensions is also singly-generated under isomorphisms and direct limits, and in particular is co-reflective. We also establish several new relations between singly-generated closed classes.

1 Introduction

In a category of $R$-modules over a certain ring $R$, the study of the closure of different classes under different operations has been a hot research topic for decades (see [15] for a thorough approach). The operations include extensions, subgroups, quotients, etc. In the case of $R = \mathbb{Z}$, we are dealing, of course, with operations in the category of abelian groups.

There is another different but related problem. Given a collection $D$ of objects in a class $C$, the smallest class that contains $D$ and is closed under certain operations is called the class generated by $D$ under these operations. When this class is equal to $C$, it is said that $C$ is generated by $D$ under the operations, and the elements of $D$ are called generators of $C$. In this situation, if $D$ consists of only one object, the class $C$ is said to be singly-generated under the operations. Now the question is clear: given a class $C$ defined...
by a certain property, is it possible to construct a generator for it? A standard reference for the topic is [1].

In this paper, we will deal with different versions of these two problems in the category of groups. Rather than from the vast literature in Module and Category Theory related to these topics, our motivation and strategy come from the field of Homotopy Theory. From the nineties, A.K. Bousfield, E. Farjoun and W. Chachólski ([4], [9], [13]) among others, developed an ambitious program whose main goal was to classify spaces and spectra in terms of classes defined by their generators and certain operations, particularly homotopy direct limits (cellular classes) and fibrations (periodicity classes). Taking account of this framework, the fundamental group functor gives a natural (but non-trivial) way to understand closed classes of groups from the same viewpoint.

We are mainly interested in three classes of operations in the class of groups, namely taking direct limits, constructing extensions from a given kernel and cokernel, and taking quotients. The relation between the closed classes under these operations has been already studied in [10] from a different point of view. We focus on singly-generated classes, and in fact our first main result is a generation result. Given a group \( A \), we denote by \( \mathcal{C}(A) \) the class generated by \( A \) under direct limits, and by \( \overline{\mathcal{C}(A)} \) the class generated by \( A \) under direct limits and extensions. Then:

**Theorem 3.7** For any group \( A \), there exists a group \( B \) such that \( \overline{\mathcal{C}(A)} = \mathcal{C}(B) \).

Moreover, we are able to describe \( B \) in terms of groups of the class \( \overline{\mathcal{C}(A)} \). This statement allows to generalize results of [18], where the case in which the Schur multiplier of \( A \) is trivial was studied, and also of [2] for the case of acyclic groups. It can also be seen as the group-theoretic version of the main theorem of [11], and it is in the spirit (see last section) of the generators of radicals described in [8].

Section 4 is devoted to establish relations between the closed classes under the operations mentioned above. The hierarchy is the following:

\[
\begin{array}{c}
\mathcal{C}(A) \to \overline{\mathcal{C}(A)} \\
\downarrow \quad \downarrow \\
\mathcal{C}_q(A) \to \mathcal{C}_t(A)
\end{array}
\]

where \( \mathcal{C}_q(A) \) is the closure of \( \mathcal{C}(A) \) under quotients and \( \mathcal{C}_t(A) \) is the closure of \( \mathcal{C}(A) \) under quotients and extensions. All these classes turn out to be co-reflective. Furthermore, we
find an example where the four classes are distinct (see Example 4.7).

The co-reflections associated to the classes $C(A)$, $C_q(A)$ and $C_t(A)$ are given respectively by the $A$-cellular cover $C_A$, the $A$-socle $S_A$ and the $A$-radical $T_A$, which are well-known notions in Group/Module Theory (see definitions in Section 2). However, for the class $C(A)$, the existence of the corresponding co-reflection $D_A$ had been previously established [18] only for groups for which there exists a Moore space, and this goal was achieved only by topological methods. This gap in the theory is filled in this paper making use of Proposition 3.7, that guarantees the existence of $D_A$ for any group $A$. In general, given a group $G$, the philosophy is the following: a singly-generated class $C$ and the co-reflection $F$ associated to a certain operation, the kernel of the homomorphism $F(G) \to G$ measures how far $G$ is from belonging to the class generated from $C$ under the operation. We prove how the co-reflections define the corresponding closed classes, describe when possible the kernels, and discuss, for different instances of the generator, the (in)equality between the corresponding classes. In particular, we prove the following:

**Proposition 4.10** Suppose that for a group $A$ the class $C(A)$ is closed under extensions, i.e. $C(A) = C(A)$. Then $C_q(A) = C_t(A)$.

The converse (stated as Question 4.12) is not true in general. Section 5 is devoted to give a counterexample that involves the Burnside idempotent radical and the Thompson group. It is an interesting question to find groups for which this converse is true. In fact, we are not even aware of any example of abelian group $A$ such that $S_A = T_A$, see Proposition 4.9 and the discussion that precedes it.

All the classes that appear in this paper will be supposed to be closed under isomorphisms.

## 2 (Co)reflections

In this section we introduce the main notions that will be used throughout the rest of our paper. We will start with the classical categorical definition of co-reflection, that will be crucial for us.

**Definition 2.1** Given a inclusion of categories $C \subseteq C'$, where $C$ is full in $C'$, $C$ is co-reflective in $C'$ if the inclusion functor possesses a right adjoint, called co-reflection.
Given a co-reflective category, the co-reflection is unique up to natural isomorphism. A good survey about the topic from the categorical point of view can be found in [1].

The following definitions will be of great interest, and will be used in particular in the last section to define some interesting co-reflective subcategories of groups.

**Definition 2.2** Let $A$ and $H$ be groups.

1. The $A$-socle $S_A H$ is the subgroup of $H$ generated by the images of all the homomorphisms $A \to H$.

2. The $A$-radical $T_A H$ of $H$ is the largest normal subgroup of $H$ such that
   $$\text{Hom}(A, H/T_A H) = 0.$$ 

3. The $A$-cellular cover of $H$ is the unique group $C_A H \in \mathcal{C}(A)$, up to isomorphism, that can be constructed from copies of $A$ by iterated direct limits, and such that there exists a homomorphism $C_A H \to H$ which induces a bijection $\text{Hom}(A, C_A H) \cong \text{Hom}(A, H)$. Observe that this is equivalent to $\text{Hom}(L, C_A H) \cong \text{Hom}(L, H)$ being a bijection for every $L \in \mathcal{C}(A)$.

**Remark 2.3** Note that similar definitions are possible if we change $A$ by a collection $\mathcal{A}$ of groups indexed over a certain set $I$. In this case, the generator would be the free product of isomorphism classes of elements of $\mathcal{A}$. Moreover, it is possible to give versions of the socle and the radical relative to a class of epimorphisms of groups. See [1] for a general treatment of these topics.

Let us recall more constructive versions of some of these groups. The $A$-radical $T_A H$ is the union $T_A(H) = \bigcup_i T^i$ of a chain starting at $T^0 = S_A(H)$ and defined inductively by $T^{i+1}/T^i = S_A(H/T^i)$, and $T^\lambda = \bigcup_\alpha T^\alpha$ if $\lambda$ is a limit ordinal. In turn, the $A$-cellular cover $C_A G$ is constructed by taking coequalizers of free products of copies of $A$, being the homomorphisms induced by coequalizers. See details in ([18], Proposition 2.1).

In the concrete case of the $A$-cellular cover, there is the following description, which is very useful:

**Theorem 2.4** ([18], Theorem 2.7) *Given groups $A$ and $H$, there is a central extension
   $$K \hookrightarrow C_A(H) \twoheadrightarrow S_A(H),$$
   such that $\text{Hom}(A_{ab}, K) = 0$ and the natural map $\text{Hom}(A, H) \to H^2(A; K)$ is trivial. Moreover, this extension is universal with respect to this property.*
The group constructions described in this section will give rise to different co-reflections, as we will see in Section 4.

We will also need to recall some basics on (co)-reflections of spaces. Analogously to groups, given a pointed space $M$, the cellular class $\mathcal{C}(M)$ generated by $M$ is the smallest class that contains $M$ and is closed under pointed homotopy colimits. The corresponding co-reflection $CW_M X \to X$ for a pointed space $X$, called the $M$-cellular cover of $X$, was explicitly described by Chachólski in his thesis [9] in the following way: consider the homotopy cofiber $\text{Cof}_M X$ of the evaluation map $\bigvee_{[M,X]} M \to X$. Denote by $P_M$ the $M$-nullification functor, i.e. the homotopical localization with respect to a constant map (see [13], chapter 1). Then the homotopy fiber of $Y \to P_{\Sigma M} \text{Cof}_M X$ is homotopy equivalent to $CW_M X$.

In [18] the relation between cellular covers of groups and spaces was described: if $M$ is a 2-dimensional complex with $\pi_1 M = A$, then $\pi_1 CW_M K(N,1) = C_A N$ for every group $N$.

### 3 Generating classes of groups

As said in the introduction, one of the goals of this paper is to describe a generator for the class $\overline{\mathcal{C}(A)}$, the smallest class that contains $A$ and is closed under direct limits and extensions. The elements of this class are usually called $A$-acyclic groups. In this section we will explain how to construct a generator for this class:

**Definition 3.1** Given a class $\mathcal{C}$ of groups closed under direct limits, a cellular generator for the class $\mathcal{C}$ is a group $B$ such that $\mathcal{C} = \mathcal{C}(B)$.

In our context, $\mathcal{C}$ will usually be the class $\overline{\mathcal{C}(A)}$ for some group $A$. These generators are not unique, and in fact several models for particular instances of $A$ have been proposed in the literature ([2], [18]). We now present a general construction, which is inspired in the methodology of Chachólski-Parent-Stanley [11], that solved successfully the same problem in the category of topological spaces. Concretely, we translate to the group-theoretic context their idea of small space; aside this, the proofs are very different, as their methods are homotopical in nature.

We start with a set-theoretic definition:

**Definition 3.2** For a given infinite regular cardinal $\lambda$ a group $H$ is called $\lambda$-small (or small with respect to $\lambda$) if $\#H$ is smaller than the cofinality of $\lambda$. 

5
Given a class \( \mathcal{C} \) of groups closed under direct limits, let \( \mathcal{C}[\lambda] \) be the subclass of groups \( H \) in \( \mathcal{C} \) such that for each \( \lambda \)-small group \( K \), every group homomorphism \( K \to H \) factors as \( K \to C \to H \) for some \( \lambda \)-small group \( C \) in \( \mathcal{C} \). The following holds:

**Proposition 3.3** Let \( \lambda \) be an infinite regular cardinal. A group \( H \) is in \( \mathcal{C}[\lambda] \) if and only if it is isomorphic to a direct limit of \( \lambda \)-small groups in \( \mathcal{C} \).

**Proof.** Suppose \( H \in \mathcal{C}[\lambda] \). Consider the direct system of all \( \lambda \)-small subgroups of \( H \) that belong to \( \mathcal{C} \), ordered by inclusion, and let \( G \) be its direct limit. We will check that \( G = H \).

By contradiction, suppose there exists an element \( a \in G \setminus H \), and call \( A \) the subgroup that it generates. The group \( A \) is \( \lambda \)-small, and hence there is a factorization \( A \to C_0 \xrightarrow{\phi_0} H \) of the inclusion \( A \to H \) such that \( C_0 \) is \( \lambda \)-small and belongs to \( \mathcal{C} \). Call \( K_0 \) the image of \( \phi_0 \), which is clearly \( \lambda \)-small. If \( K_0 = A \), then \( A \) is a retract of \( C_0 \), hence belongs to \( \mathcal{C} \) (as a retract is a direct limit) and therefore \( a \in G \), which is a contradiction, so we are done.

If not, then \( A \leq K_0 \), and we may proceed in the same way with the inclusion \( K_0 < H \), obtaining a factorization \( K_0 \to C_1 \xrightarrow{\phi_1} H \). Iterating the process if necessary, we obtain a string of homomorphisms such that for every \( i \) there is a strict inclusion \( K_i \leq K_{i+1} \):

\[
A \hookrightarrow C_0 \twoheadrightarrow K_0 \hookrightarrow C_1 \twoheadrightarrow K_1 \hookrightarrow C_2 \twoheadrightarrow K_2 \hookrightarrow \ldots,
\]

where \( C_i \) is \( \lambda \)-small in \( \mathcal{C} \) for every \( i \). As this string of arrows is a factorization of the inclusion \( A \to H \), it must eventually stabilize, and this implies that for a certain \( i \) (possibly transfinite ordinal) we have \( K_i = C_i = K_{i+1} \), and hence \( K_i \) is a retract of \( C_i \) and belongs to \( \mathcal{C} \). Again, this yields \( a \in G \), contradiction, and then \( G = H \).

Conversely, assume that \( H \) is a direct limit of \( \lambda \)-small groups in \( \mathcal{C} \), and let us prove that \( H \) belongs to \( \mathcal{C}[\lambda] \). As every direct limit of groups is a composition of limits over filtered categories and push-outs, we only need to check these two cases.

If \( H \) is the direct limit of the \( \lambda \)-small groups \( \{G_\alpha\} \) over a filtered category, a standard compactness argument states that every homomorphism \( K \to H \) factors through \( G_\alpha \) for a certain \( \alpha \). Hence \( H \) belongs to \( \mathcal{C}[\lambda] \). Now if \( H \) is a pushout of groups, the short argument of the proof of Lemma 4.8.4 in [11] applies to the group-theoretic case, and implies that \( H \in \mathcal{C}[\lambda] \). So we are done. \( \square \)

Now we can establish closure properties of \( \mathcal{C}[\lambda] \). The first is a straightforward consequence of the previous result:
Corollary 3.4 If $C$ is closed under direct limits, $C[\lambda]$ is so for every infinite regular cardinal $\lambda$.

Proof. Let $H$ be a direct limit of groups in $C[\lambda]$. By the previous proposition, $H$ is a direct limit of $\lambda$-small groups in $C$, and, every direct limit of $\lambda$-small groups in $C$ belongs to $C[\lambda]$, so we are done. □

Observe that the proof establishes that the $\lambda$-small groups in $C$ generate $C[\lambda]$ as a cellular class.

The following result is the group-theoretical analogue of [11, Theorem 6.1], although the proof is very different and purely group-theoretical.

Lemma 3.5 If $C$ is any class of groups closed under extensions, then $C[\lambda]$ is also closed under extensions.

Proof. Let $1 \to X \to Y \overset{\pi}{\to} Z \to 1$ be a short exact sequence with $X$ and $Z$ in $C[\lambda]$. We have to prove that $Y \in C[\lambda]$. Let $j : K \to Y$ be any group homomorphism, with $K$ $\lambda$-small. Since $Z \in C[\lambda]$, $\pi j$ factors through some $\lambda$-small group $Z_0$ in $C$. Taking the pullback along $Z_0 \to Z$ we obtain a short exact sequence $1 \to X \cong \ker \pi_0 \to Y_0 \overset{\pi_0}{\to} Z_0 \to 1$. Let $j_0 : K \to Y_0$ be the induced homomorphism and consider $K_0$ the subgroup of $Y_0$ generated by $j_0(K)$ and $s(Z_0)$ where $s$ is a chosen section (non necessarily homomorphism) of $\pi_0$. The intersection $K' = K_0 \cap X$ is $\lambda$-small because $K_0$ is $\lambda$-small, and it is normal in $K_0$ with $K_0/K' \cong Z_0$. Since $X \in C[\lambda]$, the inclusion $K' \hookrightarrow X$ factors as $K' \hookrightarrow X_0 \to X$ for some small group $X_0 \in C$.

Let now $P_0$ be the push-out of $K_0 \leftarrow K' \to X_0$, and denote by $(X_0)^N$ the normal closure of $X_0$ in $P_0$. The quotient $P_0/(X_0)^N \cong Z_0$, and $P_0$ is a $\lambda$-small group in $C$, because both $K'$ and $X_0$ are $\lambda$-small. Observe that $(X_0)^N$ need not be in $C$. In any case, it is $\lambda$-small and therefore the inclusion $(X_0)^N \to X$ factors through some $\lambda$-small group $X_1 \in C$. Iterating this process we get an increasing chain of short sequences over $X \to Y \to Z$ which are, alternatively, in $C$, but not exact, and exact but not in $C$. Here the subscript “$\infty$” denotes the colimit of each linear sequence of homomorphisms.
In the limit we get a short exact sequence $1 \rightarrow X_\infty = (X_\infty)^N \rightarrow P_\infty \rightarrow Z_0 \rightarrow 1$, where $X_\infty$ is $\lambda$-small and is in $\mathcal{C}$, hence $P_\infty$ is $\lambda$-small and is in $\mathcal{C}$ too, because $\mathcal{C}$ is closed under extensions. Hence, $K \rightarrow Y$ factors $K \rightarrow P_\infty \rightarrow Y$, and this shows that $\mathcal{C}[\lambda]$ is closed under extensions. \hfill \Box

For a given cardinal $\lambda$, let $B_\lambda$ the free product of a set of representatives of isomorphism classes of $\lambda$-small groups in $\mathcal{C} (A)$. We will need the following result:

Lemma 3.6 For every infinite regular cardinal, $\mathcal{C}(A)[\lambda] = \mathcal{C}(B_\lambda)$.

Proof. If $K$ is an element of $\mathcal{C}(A)[\lambda]$, it is by Proposition 3.3 a direct limit of $\lambda$-small groups in $\mathcal{C}(A)$. But every such $\lambda$-small group is a retract of $B_\lambda$, so it belongs to $\mathcal{C}(B_\lambda)$, and hence so does $K$ since $\mathcal{C}(B_\lambda)$ is closed under direct limits. Conversely, every group $H$ in $\mathcal{C}(B_\lambda)$ is an (iterated) direct limit of copies of $B_\lambda$, and in particular a direct limit of $\lambda$-small groups, so $H \in \mathcal{C}(A)[\lambda]$ and we are done. \hfill \Box

Now we are ready to prove the main theorem in this section:

Theorem 3.7 For any group $A$, there exists a group $B$ such that $\mathcal{C}(A) = \mathcal{C}(B)$.

Proof. Let $A$ be any group, $\lambda$ an infinite regular limit cardinal with $\text{cf}(\lambda) \geq \#A$. Then, by Lemma 3.6, we have $\mathcal{C}(A)[\lambda] = \mathcal{C}(B)$, where $B = B_\lambda$. In particular, the class $\mathcal{C}(A)[\lambda]$ is closed under direct limits. The conditions over $\lambda$ imply that $A$ is $\lambda$-small, and hence $\mathcal{C}(A) \subseteq \mathcal{C}(A)[\lambda]$. But by Lemma 3.5 $\mathcal{C}(A)[\lambda]$ is closed under extensions, so $\mathcal{C}(A) \subseteq \mathcal{C}(A)[\lambda]$. On the other hand, $\mathcal{C}(A)[\lambda]$ is a subclass of $\mathcal{C}(A)$ by definition, and the result follows. \hfill \Box

It is likely that the assumptions over the cardinals may be relaxed, but here we are not interested in some models of the cellular generators, but rather in their existence. In the next section we will use different cellular generators for concrete computations and in particular, and perhaps more important, to define in general the co-reflection associated to the class $\mathcal{C}(A)$ for every group $A$.

4 Closed classes

4.1 Classes of groups and associated co-reflections

Let $A$ be a group. The goal of this section is to analyze the relations between four different classes that contain the group $A$, namely:
• The cellular class $\mathcal{C}(A)$, the closure of $\{A\}$ under direct limits. Their elements are called $A$-cellular groups.

• The socular class $\mathcal{C}_q(A)$, the closure of $\{A\}$ under direct limits and quotients. Their elements are called $A$-generated or $A$-socular groups.

• The radical class $\mathcal{C}_t(A)$, the closure of $\{A\}$ under direct limits, extensions and quotients. Their elements are called $A$-constructible or $A$-radical groups.

• The acyclic class $\overline{\mathcal{C}}(A)$, the closure of $\{A\}$ under direct limits and extensions. Their elements are called $A$-acyclic groups.

The hierarchy between these classes can be depicted in the following way:

$$
\begin{array}{c}
\mathcal{C}(A) \\
\downarrow \\
\mathcal{C}_q(A) \\
\downarrow \\
\mathcal{C}_t(A)
\end{array}
\xrightarrow{\text{(4.2)}}
\begin{array}{c}
\overline{\mathcal{C}}(A) \\
\downarrow \\
\overline{\mathcal{C}}_q(A) \\
\downarrow \\
\overline{\mathcal{C}}_t(A)
\end{array}
$$

We use in particular the terminology of W. Chachólski, who studied the problem in the non singly-generated case in [10].

It is known that a group $G$ is $A$-cellular if and only if $C_A G = G$ ([18], section 2). We will check now that similar statements hold for $A$-generated and $A$-constructible groups. Probably this result is known to experts, but we do not know any explicit proof of it. We offer purely algebraic arguments here.

**Proposition 4.1** Let $A$ and $G$ be groups. Then:

• $G$ is $A$-generated if and only if $S_A G = G$.

• $G$ is $A$-constructible if and only if $T_A G = G$.

**Proof.** For the first part, it is clear that $A$ is $A$-generated. Now, if $G$ is a direct limit of $A$-generated groups, it is in fact a quotient of a free product of copies of $A$-generated groups (which is $A$-generated itself), and then it is $A$-generated. Finally, every quotient of an $A$-generated group is $A$-generated. Conversely, if $S_A G = G$, $G$ is a quotient of a free product of copies of $A$ (by definition of socle), and hence belongs to the closure of $A$ under direct limits and quotients.
Let us check the second statement. Consider the class $C$ of groups such that $G = T_A G$. Clearly $A$ belongs to $C$, so let see that $C$ is closed under quotients and extensions. For the first, recall that for every homomorphism of groups $f : H \to G$, $f(T_A H) \subseteq T_A G$. Then, if $H = T_A H$ and $G$ is a quotient of $H$, $G = T_A G$, and hence $G$ belongs to $C$. On the other hand, assume that $G$ is an extension $G_1 \to G \to G_2$, with $G_1$ and $G_2$ in $C$. As $G_1 = T_A G_1$, $G_1 \subseteq T_A G$. Hence, there is a projection $G_2 \to G/T_A G$. But $G_2 = T_A G_2$ and $T_A(G/T_A G) = 0$, thus $G/T_A G = 0$ and $G = T_A G$. So $C$ is also closed under extensions, and then $C \subseteq \mathcal{C}_A(A)$. Now assume $G = T_A G$. Then $G$ is constructed out of $A$ by means of $A$-socles, extensions and a direct union, and thus, by the previous case and the fact that a direct union is a direct limit, $G$ belongs to the closure of $A$ by direct limits, quotients and extensions. So $C \subseteq \mathcal{C}_A(A)$, and we are done.  

The previous result provides co-reflections for the respective classes.

**Corollary 4.2** In the previous notation, the categories $\mathcal{C}(A)$, $\mathcal{C}_q(A)$ and $\mathcal{C}_r(A)$ are co-reflective. The co-reflections are respectively given by the $A$-cellular cover, the $A$-socle and the $A$-radical.  

In this context, the next question is immediate: giving a group $A$, is the class $\overline{\mathcal{C}(A)}$ co-reflective? We will see next that Theorem 3.7 gives a general answer to this question. A partial solution was obtained in [18], where the authors assume the existence of a 2-dimensional Moore space for $A$ in order to construct the co-reflection. Recall that given a group $A$, a *Moore space* of type $M(A,1)$ is a CW-complex $X$ such that $\pi_1 X = A$ and $H_2 X = 0$. Then the following is proved:

**Theorem 4.3 ([18], Theorem 3.1)** Let $A$ be the fundamental group of a two-dimensional Moore space. Then the inclusion $\overline{\mathcal{C}(A)} \hookrightarrow \text{Groups}$ admits a right adjoint $D_A$. Furthermore, for each group $H$, there is a central extension

$$L \hookrightarrow D_A(H) \twoheadrightarrow T_A(H)$$

such that $\text{Hom}(A_{ab}, L) = \text{Ext}(A_{ab}, L) = 0$, and this extension is universal (initial) with respect to this property.

This result presents interesting features. First, it gives a manageable way to compute the value of the co-reflection $D_A H$ over any group $H$, as $L$ is identified as the fundamental
group of the homotopy fiber of the \( M(A,1) \)-nullification of \( K(T_AH,1) \), i.e. localization with respect to the constant map over \( M(A,1) \) (see [13], chapter 1). Moreover, the similarity with Theorem 2.4 should be remarked, and in fact for every group \( H \) there exists a natural homomorphism \( C_A^H N \to D_A^H N \) which induces a map between the corresponding extensions, and in particular the inclusion \( S_A^H N \to T_A^H N \).

Remark that the co-reflection is constructed here in a strongly non-algebraic but homotopical way. This homotopical nature of the construction made it impossible to define \( D_A^H \) for a general \( A \), as it depended on the existence of a two-dimensional Moore space for the group. We will avoid the difficulty by defining \( D_A^H \) as a cellular cover in a pure group-theoretical way.

We present now our general construction of the co-reflection \( D_A^H \) for every group \( A \).

**Proposition 4.4** Let \( A \) be a group, and \( B \) a cellular generator of the class \( \overline{C(A)} \). Then a group \( G \) is \( A \)-acyclic if and only if \( C_B^A G = G \).

**Proof.** It is a straightforward consequence of Theorem 3.7. \( \square \)

**Corollary 4.5** For every group \( A \), the class \( \overline{C(A)} \) is co-reflective, and the co-reflection is given by \( C_B^A \).

As co-reflections are unique, the value of \( C_B^A \) does not depend on the concrete model for the generator \( B \); hence, in the following this co-reflection will be denoted by \( D_A^H \). Note that there is no ambiguity in the notation, as again by uniqueness, the value of the co-reflection coincides with the one defined in Theorem 4.3, for the instances of \( A \) for which that theorem is valid. Finally, remark that our model for \( B \) in the previous section is built in a pure algebraic way, and hence it is the construction of \( D_A^H \).

The following proposition provides examples of concrete computations of \( D_A^H \), in particular when \( H_2A \neq 0 \).

**Proposition 4.6** Let \( H \) and \( G \) be finite non-trivial groups, with \( H \) simple, \( |H| \leq |G| \) and such that there is exactly a subgroup of \( G \) that is isomorphic to \( H \). Then \( D_H^G G = H \), and \( G/H \) does not belong to \( \overline{C(H)} \).

**Proof.** As \( H \) is simple and normal in \( G \), \( S_H^G G = T_H^G G = H \). But for every group \( A \) we always have \( D_A = D_A^H T_A \), and hence

\[
D_H^G G = D_H^G T_H^G G = D_H^G H = H,
\]
as $H$ is clearly $H$-acyclic. Moreover, if $G/H$ belonged to $\overline{C(H)}$, then $G$ would also belong, and that is impossible because $D_{H}G \neq G$. □

As said above, this result gives information for classes $\overline{C(A)}$ with $H_{2}A \neq 0$, a very difficult issue without a general construction of $D_{A}$. For example, if $H = A_{n}$, $G = \Sigma_{n}$ and $n \geq 5$, the proposition implies that $D_{A_{n}}\Sigma_{n} = A_{n}$, and hence neither $\Sigma_{n}$ nor $\mathbb{Z}/2$ can be constructed out of $A_{n}$ by direct limits and extensions.

### 4.2 Comparing the classes

From now on we will concentrate in measuring the difference between the classes above. Let $F$ be one of the four co-reflections. Given groups $A$ and $G$, $FG$ measures in general how close the group $G$ is from belonging to the corresponding class. Moreover, the homomorphisms $S_{A}G \rightarrow G$, $T_{A}G \rightarrow G$ and $S_{A}G \rightarrow T_{A}G$ are always injective, while the homomorphisms $D_{A}G \rightarrow T_{A}G$ and $C_{A}G \rightarrow S_{A}G$ are always surjective, so we have a commutative diagram:

\[
\begin{array}{ccc}
C_{A}G & \longrightarrow & S_{A}G \\
\downarrow & & \downarrow \\
D_{A}G & \longrightarrow & T_{A}G
\end{array}
\] (4.3)

Note that in general, $C_{A}G \rightarrow D_{A}G$ is neither surjective nor injective.

Using an example, we start by showing that the four inclusions of classes in the diagram 4.2 above are strict in general. Of course, the previous results imply that two such classes are identical if and only if their associated co-reflections are the same.

**Example 4.7** Consider the group

\[G = \langle a, b, c, d \mid a^{4} = b^{4} = c^{4} = d^{4} = 1, abab = ccdc \rangle.\]

This group is generated by order 4 elements, and hence $T_{\mathbb{Z}/2}G = G$. On the other hand, it is easy to check that the $\mathbb{Z}/2$-socle of $G$ is generated by the squares of the generators, and then

\[S_{\mathbb{Z}/2}G = \langle e, f, g, h \mid e^{2} = f^{2} = g^{2} = h^{2} = 1, efe = ghgh \rangle.\]

Observe that $S_{\mathbb{Z}/2}G$ is the group $N$ described in Example 2.11 of [18]. Hence, as $C_{\mathbb{Z}/2}G = C_{\mathbb{Z}/2}S_{\mathbb{Z}/2}G$, the cellular cover of $G$ is an extension of $S_{\mathbb{Z}/2}G$ by $\mathbb{Z}$. Finally, the same
reasoning of the mentioned example proves that $D_{\mathbb{Z}/2}G$ is an extension of $G$ by $\mathbb{Z}[1/2]$. As these four groups are different, we have that the four inclusions of classes are strict in the case $A = \mathbb{Z}/2$.

Next we will discuss when the socular class $C_q(A)$ is closed under extensions, and hence equal to the radical class $C_t(A)$. For example, if $A = \mathbb{Z}/p$ for some prime $p$, it is known that there are many groups $G$ such that $G/S_pG$ has again $p$-torsion, and same happens if we change $\mathbb{Z}/p$ by $\ast_{j \geq 1} \mathbb{Z}/p^j$; this contrasts with the case of abelian groups, in which for $A = \bigoplus_{j \geq 1} \mathbb{Z}/p^j$, $T_A N = S_A N$ for every abelian group.

When $A$ is the additive group of the rational numbers, the situation is similar. If $N$ is abelian, $S_Q N = T_Q N$, because the image of every homomorphism $\mathbb{Q} \to N$ is divisible and then splits out of $N$. However, if $N$ is not abelian the equality is not true in general. We have found no example of this situation in the literature, so we describe one in the sequel:

**Example 4.8** Let $N$ be the push-out of inclusions $\mathbb{Q} \leftarrow \mathbb{Z} \hookrightarrow \mathbb{Z}[1/p]$. As $N$ is torsion-free, every homomorphism $\mathbb{Q} \to N$ is injective. Moreover, given a subgroup $A < N$ isomorphic to $\mathbb{Q}$, $S_Q N$ is isomorphic to the subgroup generated by the conjugates of $A$ by elements of $\mathbb{Z}[1/p]$. This implies that $N/S_Q N = \mathbb{Z}[1/p]/\mathbb{Z}$, which is isomorphic to $\mathbb{Z}/p^\infty$. As there exists an epimorphism $\mathbb{Q} \to \mathbb{Z}/p^\infty$, we conclude that $N = T_Q N$, which strictly contains $S_Q N$.

It can be stated in an analogous way that $S_{\mathbb{Z}[1/p]} N \neq T_{\mathbb{Z}[1/p]} N$, as $\mathbb{Q}/\mathbb{Z}$ has non-trivial homomorphisms from $\mathbb{Z}[1/p]$; moreover, similar arguments can be used to prove that for a non-trivial set of primes $J$, the co-reflection $S_{\mathbb{Z}[J^{-1}]}$ is never equal to $T_{\mathbb{Z}[J^{-1}]}$. In particular we obtain:

**Proposition 4.9** If $A$ is an additive subgroup of $\mathbb{Q}$, $S_A = T_A$ if and only if $A$ is trivial or $A = \mathbb{Z}$.

**Proof.** It is immediate from the previous discussion and Example 5.2 of [18] (see also [14]). \[\square\]

The moral here is that it is not easy to find examples for which $S_A = T_A$. However, there are at least two ways of constructing such examples. The first family arises in the context of varieties of groups. Consider a set of equations, and the verbal subgroup $W$.
defined by these equations in the free group $F_\infty$. It is proved in [8] that for every $W$ there exists a locally free group $A$ such that the homomorphisms of $W$ into any group $G$ identify the $W$-perfect radical, and in particular, this fact guarantees that $S_A = T_A$. The inspiring (and first) “generator” of this kind was the acyclic group of Berrick-Casacuberta [2], and a closely related group will be the crucial ingredient of Proposition 5.2 below.

The second family of examples can be constructed as a byproduct of Theorem 3.7, taking account of the following two results:

**Proposition 4.10** Suppose that for a group $A$ the class $C(A)$ is closed under extensions, i.e. $C(A) = \overline{C(A)}$. Then $C_q(A) = C_t(A)$.

**Proof.** In this case, for every group $G$, $C_AG = D_AG$, and then there is a diagram

$$
\begin{array}{c}
D_AG \longrightarrow S_AG \\
\downarrow \text{id} \quad \quad \quad \downarrow j \\
D_AG \longrightarrow T_AG
\end{array}
$$

As $j$ should be an epimorphism, and it is always a monomorphism, it is an isomorphism. \hfill \square

**Corollary 4.11** Let $A$ be a group. Then for any cellular generator $B$ of the class $\overline{C(A)}$ we have $S_B = T_B$.

Recall that by Theorem 3.7, it is always possible to construct such $B$ out of any given $G$, and this provides many examples of groups for which $S_A = T_A$. It is natural and interesting to ask if there is a converse of the previous proposition, so we will devote the rest of the paper to the following problem:

**Question 4.12** If $T_A = S_A$, for a certain group $A$, is it true that $C(A) = \overline{C(A)}$?

We will see in Proposition 5.2 that the general answer is negative. Before, we will discuss in which cases the conjecture can be true. The following result is easy to prove and gives a sufficient condition:

**Proposition 4.13** Let $A$ be a group, and assume that $C(A) = C_q(A)$ and $T_A = S_A$. Then $C(A) = \overline{C(A)}$.  

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Proof. As the members of the classes are defined precisely as the groups for which the corresponding co-reflection is the identity, $T_A = S_A$ implies $C_q(A) = C_t(A)$, and then by hypothesis $C(A) = C_t(A)$. But the inclusions $C(A) \subseteq \overline{C(A)} \subseteq C_t(A)$ always hold, so the result follows.

It can be seen that the condition of the proposition holds for the acyclic group of Berrick-Casacuberta. However, the following example shows that the condition is not necessary in general for cellular generators of closures under extensions:

Example 4.14 Given a prime $p$, consider a cellular generator $B$ of the class $\overline{C(\mathbb{Z}/p)}$. If $C(B) = C_q(B)$, this would imply that $C_B = S_B$, and equivalently $D_{\mathbb{Z}/p} = T_{\mathbb{Z}/p}$. But this is not true in general, as can be deduced for instance from Example 4.7. Observe that we obtain in particular that $C(B) = \overline{C(B)} \neq C_q(B) = C_t(B)$ in this case.

We finish this section with a technical description of the difference between $D_A$ and $C_A$ when $S_A = T_A$. Remember from the previous section that for every group $N$ we have constructed $D_A N$ as the cellular cover of $N$ with respect to a cellular generator $B$ of $\overline{C(\mathbb{A})}$. As $C_A N$ is $B$-cellular, there is a canonical homomorphism $C_A N \to C_B N$, which induces in turn a diagram of extensions:

\[
\begin{array}{ccc}
K_A & \longrightarrow & C_B N \\
\downarrow & & \downarrow \\
K_B & \longrightarrow & C_A N \longrightarrow S_A N
\end{array}
\]

Here $K_A$ and $K_B$ are the corresponding kernels. Observe that $S_A N = T_A N$ implies that $S_B N = T_A N$, the right vertical homomorphism is the identity, and the difference between $C_A N$ and $C_B N(= D_A N)$ is exactly the difference between $K_A$ and $K_B$.

We will need some homotopical tools in our description of the kernel of $C_A N \to C_B N$. In the sequel we will denote respectively by $M_A$ and $M_B$ 2-dimensional complexes with $\pi_1 M_A = A$ and $\pi_1 M_B = B$, respectively, and by $\text{Cof}_A$ and $\text{Cof}_B$ the corresponding homotopy cofibres defined at the end of Section 2. Without loss of generality, we can assume that $N = T_A N = S_A N$, and denote $X = K(N, 1)$. The long exact sequence of the fibration $X \to P_{\Sigma M_A} \text{Cof}_A$ gives rise to an extension of groups:

\[
\pi_2 P_{\Sigma M_A} \text{Cof}_A \to \pi_1(CW_AX) \to N.
\]
On the other hand, Mayer-Vietoris sequence of the cofibration gives rise to another extension:

\[ H_A \to \pi_2 \text{Cof}_A \to \bigoplus L_A, \]

where the group on the left is the quotient of \( H_2 N \) under the image of the induced homomorphism \( \bigoplus H_2(A) \to H_2(N) \) and the group on the right is a subgroup of the direct sum \( \bigoplus A_{ab} \). Similar sequences exist replacing \( A \) by \( B \).

According to Section 7 of [5], the effect of \( P_{\Sigma M_A} \) over the second homotopy group of a simply-connected space is to divide by the \( A_{ab} \)-radical. As \( T_A = T_B \), we have \( K_A = \pi_2 \text{Cof}_A/T_A(\pi_2 \text{Cof}_A) \) and \( K_B = \pi_2 \text{Cof}_B/T_A(\pi_2 \text{Cof}_B) \). Moreover, as the homomorphism \( K_A \to K_B \) is induced by the map \( \text{Cof}_A \to \text{Cof}_B \) induced in turn by the inclusion \( \bigvee M_A \to \bigvee M_B \), we obtain the following:

**Proposition 4.15** \textit{In the previous notation, the kernel of the homomorphism \( C_A N \to D_A N \) is isomorphic to the kernel of the homomorphism}

\[ \pi_2 \text{Cof}_A/T_A(\pi_2 \text{Cof}_A) \to \pi_2 \text{Cof}_B/T_A(\pi_2 \text{Cof}_B). \]

Let us remark a relationship between \( \text{Cof}_A \) and \( \text{Cof}_B \). Consider \( \bigvee M_A \) and \( \bigvee M_B \), with the wedges respectively indexed by \([M_A, X]_*\) and \([M_B, X]_*\). If \( B' \) is the homotopy cofiber of the map \( \bigvee M_A \to \bigvee M_B \) induced by inclusion, there is also a cofibration sequence \( \Sigma B' \to \text{Cof}_A \to \text{Cof}_B \). As \( P_{\Sigma M_A} B' \) is contractible, ([13], 1.D.3) implies that there is a homotopy equivalence \( P_{\Sigma M_A} \Sigma \text{Cof}_A \simeq P_{\Sigma M_A} \Sigma \text{Cof}_B \). As \( \pi_3 \Sigma \text{Cof}_A = \pi_2 \text{Cof}_A \) and same happens to \( \text{Cof}_B \), this seems an interesting way to investigate the relation between \( K_A \) and \( K_B \). However, when \( H_2 A \neq 0 \), \( M_A \) cannot be a Moore space, and these nullifications are hard to compute. In the other case, Proposition 5.1 of [18] (based in turn in Section 5 of [5]) implies that certain localizations or completions of \( \pi_2 \text{Cof}_A \) and \( \pi_2 \text{Cof}_B \) are the same. See also section 5.2 of [16].

In general, the relation between the group co-reflections \( C_A \) and \( C_B \ (= D_A) \) has been investigated when \( H_2 A = 0 \), and it is well understood when \( A \) is a cyclic group or a subring of \( \mathbb{Q} \) (see for example [18]). As seen above, the latter is not useful in our context, as \( S_A \neq T_A \) for these groups. However, in the next section, we will take profit of some features of the case \( H_2 A = 0 \) to describe a counterexample for Question 4.12.
The Burnside radical and the Thompson group

In the following $A$ will denote the generator constructed in ([8], Theorem 3.3) of the Burnside idempotent radical for a prime $p$. This is a locally-free group, and then it is torsion-free. In our computations we will need its lowest ordinary homology:

**Lemma 5.1** The first homology group $H_1(A)$ is a free module over $\mathbb{Z}[1/p]$, and $H_2A = 0$.

**Proof.** As the group is locally free, $H_2A = 0$. Moreover, the construction of $A$ in the mentioned paper (as a telescope of homomorphisms that send generators to products of $p$-powers) implies that the first homology group is the direct limit of a system

$$
\mathbb{Z} \rightarrow \bigoplus_{I_1} \mathbb{Z} \rightarrow \bigoplus_{I_2} \mathbb{Z} \rightarrow \ldots,
$$

where:

- Every direct sum is taken over a finite set of indexes.
- If $j < k$, then $I_j < I_k$.
- For every $k$, the homomorphism $\bigoplus_{I_k} \mathbb{Z} \rightarrow \bigoplus_{I_{k+1}} \mathbb{Z}$ takes the $|I_k|$ components of the left group to the first $|I_k|$ of the right group in their order, and the homomorphism is multiplication by $p$.

Hence, the direct system that defines $H_1A$ is a countable direct sum of copies of the system $\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \ldots$ and the result follows. \hfill \Box

Now consider the Thompson group $F$, defined by the presentation

$$
F = \langle x_0, x_1, x_2, \ldots \mid x_i^{-1} x_n x_i = x_{n+1} \text{ for } i < n \rangle.
$$

It is known that this is an infinite simple group such that $H_2F$ is a free abelian group in two generators, see [7].

**Proposition 5.2** For the group $A$, one has $S_A = T_A$, but $D_A \neq C_A$.

**Proof.** As $A$ generates the idempotent radical associated to a variety of groups, $T_A = S_A$. We will check in this proof that $D_A F \neq C_A F$.

First we will compute explicitly $D_A F$. As the radical is always a normal subgroup and $F$ is simple and not equal to its exponent $p$ reduction, $F = T_A F$. According to
Theorem 4.3, we need to compute the kernel of the homomorphism $D_A F \to F$, which is in turn the fundamental group of the homotopy fibre of the map $K(F, 1) \to P_M K(F, 1)$, being $M = M(A, 1)$ the corresponding 2-dimensional Moore space (which exists by the local freeness of $A$). We can describe nullifications of nilpotent spaces, so we consider the evaluation $\bigvee_{[M,K(F,1)]} M \to K(F,1)$ extended over all the homotopy classes of maps $M \to K(F,1)$. Let $C$ be the homotopy cofibre of this map. As the wedge is $M$-acyclic, $P_M K(F,1) = P_M C$. As $F = T_A F$, $C$ is simply-connected.

Now we need to compute $\pi_2 C$, which by Mayer-Vietoris is defined by an extension:

$$H_2 F \hookrightarrow \pi_2 C \twoheadrightarrow \bigoplus \mathbb{Z}[1/p].$$

Here we have used that $H_2 A = 0$ and the perfectness of $F$. Now according to ([5], Theorem 7.5), $\pi_2 P_M C$ is the Ext-$p$-completion of $\pi_2 C$. In the sequel we will call $P$ the group in the right-hand side of the extension.

Taking account of the fact that $\text{Hom}(\mathbb{Z}/p^\infty, G) = 0$ if $G$ is free abelian and ([6], VI.2.5), we obtain an extension:

$$\text{Ext}(\mathbb{Z}/p^\infty, \bigoplus \mathbb{Z}[1/p]) \hookrightarrow \text{Ext}(\mathbb{Z}/p^\infty, \pi_2 C) \twoheadrightarrow \text{Ext}(\mathbb{Z}/p^\infty, \mathbb{Z} \oplus \mathbb{Z}).$$

As the Ext-$p$-completion of every $p$-divisible group is trivial, the term in the left is so, and the homomorphism in the right is an isomorphism. Hence by ([6], VI.2.6) we have $\pi_2 P_M C = \mathbb{Z}_{p^\wedge} \oplus \mathbb{Z}_{p^\wedge}$, and $D_A F$ is defined by an extension:

$$\mathbb{Z}_{p^\wedge} \oplus \mathbb{Z}_{p^\wedge} \hookrightarrow D_A F \twoheadrightarrow F.$$ 

Now by Theorem 2.4, we must prove that the kernel of $C_A F \to F$ cannot be isomorphic to two copies of the $p$-adic integers. We know that such kernel is isomorphic to $\pi_2 C/T_A \pi_2 C$. As $\pi_2 C$ is abelian, $T_A \pi_2 C = T_{\mathbb{Z}[1/p]} \pi_2 C$, and as the homomorphism $f : \pi_2 C \to P$ is an epimorphism, same is true for $\pi_2 C/T_{\mathbb{Z}[1/p]} \pi_2 C \to P/f(T_{\mathbb{Z}[1/p]} \pi_2 C)$. But as $\mathbb{Z}[1/p]$ is $p$-divisible, $f(T_{\mathbb{Z}[1/p]} \pi_2 C)$ is so, and then it is a free $\mathbb{Z}[1/p]$-module of $P$, as $\mathbb{Z}[1/p]$ is a principal ideal domain. Then it is easy to check that no non-trivial quotient of a free $\mathbb{Z}[1/p]$-module by a $\mathbb{Z}[1/p]$-free submodule can be divisible.

Assume now by *reductio ad absurdum* that $\pi_2 C/T_{\mathbb{Z}[1/p]} \pi_2 C$ is isomorphic to $\mathbb{Z}_{p^\wedge} \oplus \mathbb{Z}_{p^\wedge}$. Then $P/f(T_{\mathbb{Z}[1/p]} \pi_2 C)$ should be non-trivial, otherwise $\mathbb{Z}_{p^\wedge} \oplus \mathbb{Z}_{p^\wedge}$ would be isomorphic to a free abelian group. Moreover, as $\mathbb{Z}_{p^\wedge} \oplus \mathbb{Z}_{p^\wedge}$ is $q$-divisible for every $q \neq p$, its quotient $P/f(T_{\mathbb{Z}[1/p]} \pi_2 C)$ should be too. But $P/f(T_{\mathbb{Z}[1/p]} \pi_2 C)$ is $p$-divisible but not divisible, so we have a contradiction, and then $D_A F \neq C_A F$. \hfill \Box
Remark 5.3 We do not know if $\pi_2 C/T_{Z[1/p]} \pi_2 C$ is a free abelian group in two generators, because we are unable to show if the extension that defines $\pi_2 C$ splits or not. Recall that $\text{Ext}(Z[1/p], Z) = Z_p^\wedge / Z$ is uncountable, but only the trivial extension of $Z$ by $Z[1/p]$ splits.

Acknowledgments. We warmly thank Jérôme Scherer for several discussions about the paper and for his thorough revision of the manuscript, which has led to a substantial improvement of the presentation. We also thank Fernando Muro for useful discussions. The first author expresses his gratitude to the Department of Mathematics of the University of Almería for their kind hospitality.

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