Assessment of groups in a network organization based on the Shapley group value☆

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A B S T R A C T

The focus of this paper is the assessment of groups of agents or units in a network organization. Given a social network, the relations between agents are modeled by means of a graph, and its functionality will be codified by means of a cooperative game. Building on previous work of Gomez et al. (2003) for the individual case, we propose a Myerson group value to evaluate the ability of each group of agents inside the social network to achieve the organization’s goals. We analyze this centrality measure, and in particular we offer several decompositions that facilitate obtaining a precise interpretation of it.

1. Introduction

The use of formal and informal social networks in an organization, as well as employees participation in virtual communities of practice for seeking knowledge, has received growing attention in the last years (see for instance [5,6]). As a direct consequence organizations are interested in using these social networks for their own interests: to promote collaborative working groups, to diffuse innovations and ideas, to force the approval of a proposal, to foster the sharing of information and knowledge to meet their business needs, etc.

Let us think for instance of a network within a consulting company as that analyzed in Borgatti [1], which consists of advice-seeking ties among members of a global company. In this framework, the identification of a small group of actors who are able to lead the formation of optimal working teams, or whose deletion would disrupt most the ability of the social structure to form them, is crucial for the top managers of the organization. In the case of virtual communities of practice, a firm can be interested in selecting a small group of its own workers that would maximize the spreading of information through the network to meet their business objectives. It could also be the case that a big firm made up of many different inter-organizational independent subsidiary firms wants to force the adoption of a certain guideline which has to be passed by the majority of its affiliates, which are managed by their own boards of corporate directors. In that case, the firm’s goal is to choose a small group of directors within each affiliate in order to achieve the required support for its guideline, taking into account the affinities between the corporate members and subsidiary firms.

The literature on network theory describes a large number of structural centrality measures (degree, betweenness or closeness, for instance) to assess the importance of each individual in a given social network (see the review of [11]). However, as Kiss and Bichler [14] point out “a critical question is which of these measures is best to select a group of agents to achieve a specific goal”. Their comment has two implications for our purposes. On the one hand, the questions stated above are related with group centrality, rather than with individual centrality. Therefore, since the k more central agents (from an individual point of view) do not form in general the most valuable group of k-agents (the ensemble issue in words of Borgatti [1]), we need a group measure that adjusts the sum of centralities so as to account for the possible complementarity and substitutability of relations among group’s members. On the other hand, none of those structural centrality measures takes into account the purpose of the organization, which is not the same in each of the examples described above, and must be incorporated to the problem at hand to measure the importance of a group in relation with the goal that is being pursued.

Considering the previous two aspects, we propose to extend to groups the individual game theoretic centrality measure introduced by Grofman and Owen [13] and Gomez et al. [12]. To be specific, given a social network, we rely on Game Theory and assume a transferable utility game (see definition in Section 2 below) that reflects the interests of the organization. Then we introduce the Myerson group value, which

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is based on the Shapley group value [10], to assess the performance of a group when the communication structure and the organization’s goal are simultaneously considered. The present paper is focused on validating from a theoretical point of view the practical application of this centrality measure, and in particular interpreting it in each context. In this sense, the decompositions of Section 5 are remarkable, as they permit to understand better the meaning of the group values; and also our results concerning the search for the best partner, because it is a very important problem to identify profitable partnership relations inside a corporation or business company.

The remainder of the paper is organized as follows. In Section 2 we introduce some standard concepts and notation on Game Theory and Graph Theory that will be used throughout this paper, and we give a brief description of the individual centrality index [12] on which our proposal is based. In Section 3 we introduce the key concepts of Myerson group value and game theoretic group centrality. Sections 4 to 6 delve into the behavior of this complex measure. In Section 4 we prove that the proposed Myerson group value simultaneously captures the potential of a group to achieve a given goal, as well as to force its underachievement. Sections 5 and 6 offer several decompositions of the proposed group value that allow to obtain a precise interpretation of it by means of evaluating the different elements that contribute to the assessment of groups in a network organization. In Section 7 we illustrate how to use empirically the methodology we propose using the network within a consulting company analyzed in Borgatti [1]. We explore how to obtain the information to construct the theoretical model, how to use the group valuation to select pertinent groups and how to gain more information about the elements that constitute the centrality of a given group using the previously derived theoretical decomposition results.

2. Preliminaries

We first recall some standard notation and concepts from game theory and graph theory that will be used throughout the paper.

A cooperative game in coalitional form with side payments, or with transferable utility (TU game), is an ordered pair \((N, v)\), where \(N\) is a finite set of players and \(v: 2^N - \mathbb{R}\), with \(2^N = \{\{S | \emptyset \subseteq S \subseteq N\}\}\), is a characteristic function on \(N\) satisfying \(v(\emptyset) = 0\). For any coalition \(S \subseteq N\), \(v(S) \in \mathbb{R}\) is the worth of coalition \(S\) and represents the reward that coalition \(S\) can achieve by itself if all its members act together. For brevity, throughout the paper, the cardinality of sets (coalitions) \(N\) and \(S\) will be denoted by appropriate small letters \(n\) and \(s\), respectively. Also, for notational convenience, we will write singleton \([i]\) as \(i\), when no ambiguity appears. A game \((N, v)\) is super-additive when \(v(S) + v(T) \geq v(S \cup T)\) for every pair of disjoint coalitions \(S, T \subseteq N\). If \(v(S) = v(T) = v(S \cup T)\) for every pair \(S, T \subseteq N\), then the game is convex. A convex game can also be characterized by \(v(S) = v(T)\) for every \(S \subseteq T \subseteq N\). A game \((N, v)\) is symmetric when all agents play the same role in the cooperative situation and thus the only relevant information is the size of the coalition which is formed. In this case, the characteristic function \(v(S)\) is defined by a real function \(f: \mathbb{R} \rightarrow \mathbb{R}\) such that \(f(0) = 0\) and \(v(S) = f(|S|)\) for all non-empty \(S \subseteq N\).

A game \((N, u)\) is a simple game if \(u(S) \in \{0, 1\}\) for all \(S \subseteq N\), and it is monotonic if \(u(S) \leq u(T)\) whenever \(S \subseteq T\). Simple games just make a fixed player set \(N\) and identify \((N, u) \in \mathcal{G}_N\) with its characteristic function \(u\) when no ambiguity appears.

An allocation \(\phi\) for TU games is an assignment which associates to each game \((N, v) \in \mathcal{G}_N\) a vector \(\phi(N, v) \in \mathbb{R}^N\), where \(\phi_i(N, v) \in \mathbb{R}\) represents the value of player \(i \in N\). Shapley [18] defines his value as follows:

\[
\phi_i(N, v) = \sum_{S \subseteq N \atop |S| = n-1} \frac{(n-1)!}{n!} (v(S \cup \{i\}) - v(S)), i \in N.
\]

The value \(\phi_i(N, v)\) of each player, which is a weighted average of his marginal contributions, admits different interpretations, such as the payoff that player \(i\) receives when the Shapley value is used to predict the allocation of resources in multiperson interactions, or his power when averages are used to aggregate the power of players in their various cooperation opportunities. Note that the difference \(v(S) - v(S_i)\) measures the ability of player \(i\) to change the worth of coalition \(S\) in case he/she joins it. Following the question originally addressed by Shapley in his seminal paper, we interpret the value as the expectations of a player (group) in a game \((N, v)\). In other words, \(\phi_i(N, v) \in \mathbb{R}\) is an a priori value that measures the prospects of player \(i \in N\) in the game \(v \in \mathcal{G}_N\), and can be used as an objective function for selecting key players.\(^1\)

Since the Shapley value is symmetric\(^2\) and efficient\(^3\) then \(\phi_i(N, v) = \frac{1}{|N|}\) for every player \(i \in N\) and for every symmetric game \((N, v)\). The Shapley value admits an alternative expression in terms of the Harary divides of every coalition \(T\) in \((N, v)\), which are given by \(\Delta^N(v, T) = \sum_{S \subseteq T} (-1)^{|S|} v(S)\). The formula is based on the linearity of the Shapley value and the expression of the game \((N, v)\) in terms of the basis of the unanimity games\(^4\): \(\phi_i(N, v) = \sum_{S \subseteq N \atop i \in S} \frac{\Delta^N(v, S)}{s}, i \in N\) (see [18]).

An undirected graph or simply a graph \((N, \Gamma)\) consists of a finite set \(N = \{1, \ldots, n\}\) of nodes and a set \(\Gamma\) of edges whose elements are unordered pairs of distinct nodes. If \((i, j)\) is an edge in \(\Gamma\), then \(i\) and \(j\) are said the endpoints of the edge, and \(i\) and \(j\) are called adjacent nodes in \(\Gamma\). The degree of a node is the number of its adjacent nodes. A graph \((N, \Gamma)\) is a subgraph of \((N, \Gamma')\) if \(N \subseteq N\) and \(\Gamma \subseteq \Gamma\), where the edge \((i, j)\) can be in \(\Gamma\) only if \(i\) and \(j\) are in \(N\). We say that \((N, \Gamma)\) is the subgraph induced by \(N\) if \(\Gamma\) contains each edge of \(\Gamma\) with endpoints in \(N\). A path between two nodes \(i\) and \(j\) in a graph \((N, \Gamma)\) is a subgraph of \((N, \Gamma)\) consisting of a sequence of nodes and edges \(P(i, j) = \{i = i_1, i_2, \ldots, i_k, j\}\), with \(k \geq 2\) satisfying the property that for all \(1 \leq s < k, 1, i_1, i_k \in \Gamma\). A cycle is a path \(P = \{i = i_1, i_2, \ldots, i_k, i\}\) where \(i\) is adjacent to every pair \(i, j \in N\) of its nodes is connected, i.e., if there is a path in the graph from node \(i\) to node \(j\); otherwise, the graph is disconnected. The maximal connected subgraphs of a disconnected graph are called its connected components, or components for short. Let \(\mathcal{S}_N\) be a subset of nodes, then \(\mathcal{C}(\mathcal{S}_N)\) will denote the set of connected components of the subgraph \((S, \Gamma_S)\) induced by \(S\). We will refer to \(\mathcal{C}(\mathcal{S}_N)\) as the set of connected components of \(S\) in \(\Gamma\). A tree is a connected graph that contains no cycle. Note that every two nodes of a tree are connected by a unique path. In this case, when \((N, \Gamma)\) is a tree, then for every \(\mathcal{S}_N\) there exists a unique smallest connected subgraph in \(\Gamma\) which contains the subgraph \((S, \Gamma_S)\), which we will call its connected hull and denote by \(H(S)\).\(^5\)

The same occurs when \((N, \Gamma)\) is a cycle-complete connected graph; recall that the term cycle-complete graph was introduced by van den Nouweland and Born [16] to refer to those graphs in which for every cycle of distinct elements \(N' = \{i_1, i_2, \ldots, i_k, \ldots, i_l, 1\}\) holds that the subgraph induced by \(N'\) is a complete graph (i.e., all nodes are adjacent to the remaining ones). For a general connected graph \((N, \Gamma)\), let \(\mathcal{M}_I(S) = \{S_1, \ldots, S_t\}\) be the set of minimal connection sets of \(S\) in \(\Gamma\) and \(\mathcal{M}_I(S) = \{S_1, \ldots, S_t\}\) be the set of agents in \(\mathcal{M}_I(S)\).

\(^1\) Note that in general \(v(i) \neq \phi_i(N, v)\).

\(^2\) \(\phi_i(N, v) = \phi_i(N, v)\) for all symmetric players \(i, j \in N\) (i.e., \(v(S) = v(T)\) for all \(S \subseteq T\)).

\(^3\) \(\Delta^N(v, T) = 0\) for every TU game \((N, v)\).

\(^4\) Recall that a game \((N, v)\) is a unanimity game if there exists a coalition \(S\) such that for every \(\mathcal{C}(\mathcal{S}_N)\), \(\mathcal{C}(\mathcal{S}_N)\) is 1 if \(\mathcal{S}_N\) and \(\Gamma = 0\) otherwise; in this case, we will denote the game by \((N, u^*\) and its Shapley value is \(\phi_i(N, u^*) = \frac{1}{|N|}\) for all \(i \in S\) and \(\phi_i(N, u^*) = 0\) otherwise. Unanimity games are a basis of the vector space \(\mathcal{G}_N\).

\(^5\) Formally, the subgraph which contains \((S, \Gamma_S)\) is the subgraph induced by \(H(S)\).
3. Assessment of groups: Myerson group value and centrality

In this section we introduce our main ideas. We propose to assess a group by means of the value of that group in the graph restricted game \((N,v_G)\), which we base on the extension of the Shapley value from individuals to groups considered in Flores et al. [10]. We also show some interesting properties of the proposed group measures.

Given a TU-game \((N,v)\) and a coalition \(C\in\mathcal{N}\), Derks and Tijj [8] define the merging game \((N,v_C)\), in which all the agents of \(\mathcal{C}\) are replaced by a single "super-agent" \(c\), who can act as a proxy of any player in \(C\). Formally, in the merging game \((N,v_C)\), the player set \(N\) will become \(N = (N\setminus C)\cup\{c\}\) with \(c\) as a single player \(c\in C\), and the characteristic function \(v_C\) describing the new situation is of the form:

\[
v_C(S) = \begin{cases} v(S), & \text{if } c \in S, \forall S \subseteq \mathcal{N}\setminus C. \\ v(SUC), & \text{if } c \in S, \exists S \subseteq C. \end{cases}
\]

Now we define the Shapley value group of the game \((N,v)\in\mathcal{G}_N\) as the mapping \(\phi^S\) that assigns to every non-empty \(\mathcal{C}\subseteq\mathcal{N}\) a real number \(\phi^S(C,N,v)\) given by

\[
\phi^S(C,N,v) := \phi(C\setminus\{c\},v_C), \text{ and being } \phi^S(\emptyset,N,v) = 0.
\]

Here \(\phi^S(C,N,v)\in\mathbb{R}\) is the value of group \(C\) and measures the prospects of group \(C\) in \(v\) when players in \(C\) act as a unit.

Note that when considering the previous merging game for evaluating the expectations of group \(C\) in the game, we do not need to suppose necessarily that the agents know each other nor agree to act jointly; instead, we may assume the existence of an external decision maker – the organization – that is able and willing to compute the value of the different groups.

The concept of group value is somewhat related with that of union value [2]. In fact, it is a restricted union value in the sense that it is defined only for partitions of players of the form \(\mathcal{P}_C = \{C,\{i\}, i \in C\}\). But, it is also an extended union value in the sense that, when partition \(\mathcal{P}_C\) is considered, the individual values of the remaining players \(i \notin C\) are not determined.

Remark 1. Note that considering \(\phi^S(C,N,v)\) as the worth of a coalition, the Shapley group value assigns another TU game to any other TU game. This new game is not interesting from the point of view of this article, and we will not discuss it henceforth.

Now, in order to measure the value of each group when the restrictions in cooperation due to the social relations are considered, we introduce the Myerson group value as the Shapley group value of the graph-restricted merging game.

Definition 1. Let \((N,\mathcal{G})\) be a social network, and \((N,v)\) a TU game. Then for every group \(\mathcal{C}\subseteq\mathcal{N}\) the graph-restricted merging game \((N_C,v_{\mathcal{C}})\) is defined as:

\[
v_{\mathcal{C}}(S) = v_{\mathcal{C}}(S) = \sum_{T_k \in \text{con}(S)} v(T_k), \text{ and } v_{\mathcal{C}}(\mathcal{SUC}) = v_{\mathcal{C}}(\mathcal{SUC}) = \sum_{T_k \in \text{con}(\mathcal{SUC})} v(T_k),
\]

for every coalition \(\mathcal{S}\subseteq\mathcal{N}\setminus C = N\setminus\{c\}\).

Definition 2. Let \((N,\mathcal{G})\) be a social network, and \((N,v)\) a TU game. Then the Myerson value of the group \(\mathcal{C}\subseteq\mathcal{N}\) is defined to be \(\phi^M(C,N,v) = \phi^N(N_C,v_{\mathcal{C}})\), for every group \(\mathcal{C}\subseteq\mathcal{N}\).

The Myerson value of \(C\) can be interpreted as an a priori valuation of the expectation of group \(C\) in the game \((N,v)\) when communications between the players are restricted by the graph \((N,\mathcal{G})\).

Now, following Gomez et al. [12] approach, we must account for the variations in value due to their position in the graph.

Definition 3. Let \((N,\mathcal{G})\) be a social network, and \((N,v)\) a TU game. Then, the centrality of the group \(\mathcal{C}\subseteq\mathcal{N}\) is defined to be:

\[
\gamma^C(N,v,\mathcal{G}) := \phi^M(C\setminus\{c\},v_{\mathcal{C}}) - \sum_{i \in C} \phi^M(N,v), \forall C \subseteq \mathcal{N}.
\]

The above differences (4) represent the increase (or decrease) in the value of group \(C\) due to two factors: its position in the graph, i.e. its positional value, and the synergies among their agents, i.e. their integration effect— which depend only on the worth of each group of agents for the purpose of the C can be reformulated as

\[
\gamma^C(N,v,\mathcal{G}) = \left(\phi^M(C\setminus\{c\},v_{\mathcal{C}}) - \phi^M(C,N,v)\right) + \left(\phi^M(C,N,v) - \sum_{i \in C} \phi^M(N,v)\right)
\]

The first difference \(\phi^M(C\setminus\{c\},v_{\mathcal{C}}) - \phi^M(C,N,v)\) measures the variation in the value of group \(C\) due to their position in the social network, whereas the second difference accounts in turn for the benefits derived from their agreement to act jointly taking into account the purpose of the organization.

Definitions 2 and 3 provide us with two measurements of value and centrality which take into account the ensemble issue and also the purpose of the organization, which is modeled by means of a TU game. In this paper we use some examples of games, although the definition of the characteristic function to be used in a specific situation should be subject to a thorough analysis. We want to remark the versatility of the game theoretic measures of centrality, since each characteristic function \(v\) may provide a different centrality measure that takes into account the specific purpose described by \(v\).

4. Positive and negative group valuation

Next, we will show that the proposed Myerson group value simultaneously measures the potential of a group to achieve the specific goal that is being pursued, as well as to force its underachievement. First we recall the definition of the dual game and the property of self-duality property of the Shapley value.

Definition 4. Let \((N,v)\in\mathcal{G}_N\) be a TU game. Then its dual game, denoted by \((N,v^*),\) is defined for all \(\mathcal{S}\subseteq\mathcal{N}\) by \(v^*(\mathcal{S}) = v(\mathcal{N}\setminus \mathcal{S})\).
The worth \(v^D(S)\) of \(S\triangleq N\) in the *dual game* measures the losses of the grand coalition if coalition \(S\) leaves the game \((N,v)\). That is, \(v^D(S)\) is the benefit that their members can block in situation when no member leaves the society, then the *dual graph-restricted* game, which we will denote by \((N,v^D)\), must be defined as follows:

\[
v^D(S) := (v^D)\big((N)\big) - v^D(N,S), \quad \forall S \subseteq N. \tag{5}
\]

The worth of coalition \(S\) in the dual game is then \(v^D(S)\) when the social network \((N,\Gamma)\) is connected. Otherwise, the graph-restricted game is an additive game over its connected components, and the dual operator applies to each of these components.

The Shapley value is *self-dual*, that is, the Shapley value of the game \((N,v)\) equals the Shapley value of its dual, and thus it simultaneously captures the positive and the negative value of every player. Next proposition shows that the same remains true at a group level.

**Proposition 1.** Let \((N,\Gamma)\) be a social network, and let \((N,v)\) be a general TU game reflecting its functionality. Then, the Myerson group value and the game theoretic group centrality are self-dual, i.e., \(\phi^G(N,v^D) = \phi^G(N,v)\) and \(\gamma^G(N,v^D) = \gamma^G(N,v)\), for every group \(C\subseteq N\).

All the proofs here and below are postponed to the Appendix, not to interrupt the natural flow of the arguments.

### 5. Myerson group value decomposition: communication and betweenness

The main goal of this paragraph is to give a general decomposition of the Myerson group value (see *Propositions 2 and 3* below) in two kinds of value – *communication* and *betweenness* – along the lines of Gomez et al. [12]. Next lemmas, that allow to express \(\phi^G(C,N,v)\) in terms of the Harsanyi dividends of the original game \((N,v)\), are needed for obtaining such decomposition. Recall that the connected hull \(H(C)\) of a subset \(C\subseteq N\) and the set \(\mathcal{L}(C)\) of its minimal connection sets is defined in *Section 2*.

**Lemma 1.** Let \((N,\Gamma)\) be a social network, and let \((N,v)\) be a TU game. If \((N,\Gamma)\) is a connected graph, then

\[
\phi^G(C,N,v_C) := \phi_C(N_C,v_C) := \sum_{S \in \mathcal{L}(C) \cap H(C)} \frac{\Delta^N(v,S)}{|N_C/H(C)[S]|},
\]

where \(N_C/S \subseteq N\) is the coalition

\[N_C/S = \begin{cases} \{S \setminus C \cup \{c\} \} & \text{if } S \cap C \neq \emptyset, \\ S \setminus C & \text{if } S \cap C = \emptyset. \end{cases}\]

**Lemma 2.** Let \((N,\Gamma)\) be a social network, and let \((N,v)\) be a TU game. If \((N,\Gamma)\) is a general connected graph \((N,\Gamma)\), let \(\mathcal{L}(S) = \{S_1, \ldots, S_k\} \neq \emptyset\) be the set of minimal connection sets of \(S\) in \(\Gamma\). Then we have

\[
\phi^G(C,N,v) := \phi_C(N_C,v_C) := \sum_{S \in \mathcal{L}(C) \cap H(C)} \Delta^N(v,S) \phi_C(N_C,u^G_C), \tag{7}
\]

where \(\mathcal{L}(S) = \bigcup_{i=1}^k S_i\) is the set of agents in \(\mathcal{L}(S)\) and \(\phi_C(N_C,u^G_C)\) is the simple game with minimal winning coalitions set \(\mathcal{M}(N_C,u^G_C) = \{N_C/S_1, \ldots, N_C/S_k\}\) for all \(S \subseteq N\).

5.1. Communication and betweenness redundancy

We formalize the concept of *redundancy* between groups of agents and then we use the above decomposition to analyze with the aid of an example how the proposed group measure accounts for the two kinds of redundancy considered by Borgatti [11]: redundancy with respect to adjacency and distance, and redundancy with respect to bridging.

**Definition 5.** Let \((N,\Gamma)\) be a social network, and \((N,v)\) be a TU game reflecting the organization goal. Then, the *redundancy* between groups \(C_1\) and \(C_2\) is defined to be:

\[
\text{Red}(C_1, C_2, N, v, \Gamma) := \gamma^G(C_1, N, v, \Gamma) + \gamma^G(C_2, N, v, \Gamma) - \gamma^G(C_1 \cup C_2, N, v, \Gamma),
\]

for every pair of groups \(C_1, C_2 \subseteq N\). Along the same lines, their *relative redundancy* is defined as the quotient

\[
\text{RRRed}(C_1, C_2, N, v, \Gamma) := \frac{\text{Red}(C_1, C_2, N, v, \Gamma)}{\phi^G(C_1, \Gamma) + \phi^G(C_2, \Gamma)}. \tag{8}
\]

Note that negative redundancies must be interpreted as a positive feature. If \(\text{Red}(C_1, C_2, N, v, \Gamma) < 0\), then groups \(C_1\) and \(C_2\) strengthen each other, i.e., they are complements.

In the following example we omit some of the algebraic manipulations because they are tedious and technical. They are available upon request.
**Example 1.** Let us consider the following social network, and let \((N, v)\) be a symmetric TU game:

Now let us consider the group \(C = \{5, 6\};\) then the redundancy of agents 5 and 6 (see **Definition 5**) is given by \(\text{Red}(\{5\}, \{6\}, N, v, \Gamma) = \phi_0(N, v, \Gamma) + \phi_0(N, v, \Gamma) - \phi_0(N, \Gamma)\).

If we apply formula (9) to the computation of \(\phi_0(N, v, \Gamma)\), we can see the difference between two kinds of redundancy of 5 and 6: with respect to communication and with respect to betweenness. The variation of communication value when agents 5 and 6 form a group accounts for their redundancy with respect to adjacency and distance, which we will call communication-redundancy, whereas the variation of betweenness value accounts for their redundancy with respect to bridging, which we will call betweenness-redundancy, i.e.,

\[
\text{Red}(\{5\}, \{6\}, N, v, \Gamma) = \text{ComRed}(\{5\}, \{6\}, N, v, \Gamma) + \text{BetRed}(\{5\}, \{6\}, N, v, \Gamma).
\]

The communication-redundancy of agents 5 and 6, using the first summand of expression (9) and after some calculations, is given by:

\[
\text{ComRed}(\{5\}, \{6\}, N, v, \Gamma) = \frac{1}{6} \Delta^N(v, \{5, 6\}) + \sum_{\sigma \in \text{CS}(\{5, 6\})} \frac{|H_1(\sigma)| - 2}{|H_1(S)|(|H_1(S)| - 1)} \Delta^N(v, S), \quad \sigma \geq 0. \tag{10}
\]

Analogously, the betweenness-redundancy of agents 5 and 6, using the second summand of expression (9), is given by

\[
\text{BetRed}(\{5\}, \{6\}, N, v, \Gamma) = \sum_{\sigma \in \text{CS}(\{5, 6\})} \frac{2}{k_1 + k_2 + 1 - k_1 k_2 + 2} \Delta^N(v, S), \quad \sigma \geq 0. \tag{11}
\]

where \(k_1 = |H_1(S_1 U \{1\})|\) and \(k_2 = |H_1(S_2 U \{7\})|\). Thus, as the coefficients in (10) are positive and the coefficients in (11) are negative, we may state that in this case agents 5 and 6 are redundant for spreading purposes, although both are strictly necessary if the goal is to break the communications. The above coefficients only deal with the structure of the network, but not with the interest on forming a coalitions. Such interests, measured through the Harssanyi dividends of coalition \(S\) in (10), and those of \(S_1 U S_2\) in (11), also determine the amount of positive and negative redundancy, respectively.

Now, if the purpose of the organization is to transmit information through bilateral communications, then the messages game gives a good model. The characteristic function of the messages game is given by \(v^{\text{msg}}(S) = 2s^2 - s, \forall S \in N\), and focuses on the level of communication activity between pairs of individuals when the benefit of a binary communication is independent of the actors performing, is the same for each relation, and communication is possible in both ways. Since the Harssanyi dividends of the messages game verify \(\Delta^N(v^{\text{msg}}, S) = 2, \forall S \in N\), and \(\Delta^N(v^{\text{msg}}, S) = 0\):

\[
\text{ComRed}(\{5\}, \{6\}, N, v^{\text{msg}}, \Gamma) = \frac{1}{6} \Delta^N(v^{\text{msg}}, \{5, 6\}) + 0 = \frac{1}{3},
\]

\[
\text{BetRed}(\{5\}, \{6\}, N, v^{\text{msg}}, \Gamma) = \frac{1}{3} - \frac{2}{4} \Delta^N(v^{\text{msg}}, \{1, 7\}) + \frac{1}{4} - \frac{2}{5} \left( \sum_{i \in S} \Delta^N(v^{\text{msg}}, \{i, 7\}) \right) + \frac{10}{4} \Delta^N(v^{\text{msg}}, \{1, j\})
\]

\[
= 68/15.
\]

Therefore agents 5 and 6 are redundant for spreading purposes, but they are complementaries with respect to bridging.

Previous decomposition results only deal with connected social networks. Nevertheless, they are general enough since the Myerson value of a group in a disconnected graph can be obtained as the sum of the Myerson value of its subgroups, when they are considered as a group in the different connected subgraphs to which each subgroup belongs.

**Proposition 4.** Let \((N, \Gamma)\) be a social network, and let \((N, v)\) a TU game. If \((N, \Gamma)\) is a disconnected graph, let \((N_k, \Gamma_k), k = 1, \ldots, r\) be its connected components. Then,

\[
\phi_k(N, v, \Gamma) = \sum_{i \in N_k} \phi_k(N_i, v, \Gamma),
\]

where \(\phi_k(N_i, v, \Gamma)\) is the restriction of \(\phi(N, v, \Gamma)\) to \(N_k \in C\). They are said to be strategic complements and substitutes, which in turn rely on the second-order difference operator for a pair of agents \(i, j \in N\).

The second-order difference operator for a pair of players \(i, j \in N\) is defined as a composition of marginal contribution operators (i.e., first-order difference operators) as follows:

\[
\partial_i^2 v(S, N, v) = v(S U \{i, j\}) - v(S U \{i\}) - v(S U \{j\}) + v(S).
\]

Here \(\partial_i^2 v(S, N, v)\) expresses player \(i\)‘s effort over the marginal contribution of player \(j\) (or vice versa). Note that \(v(S U \{i, j\}) - v(S) = \partial_i^2 v(S, N, v) + \partial_j^2 v(S, N, v) + \delta_i^2 v(S, v)\), and thus \(\partial_i^2 v(S, N, v) = 0\) implies that the marginal contribution of player \(i, j\) as a group exceeds the sum of the individual marginal contributions of each player. Therefore, \(\partial_i^2 v(S, N, v)\) can be interpreted as a measure of players \(i, j\) complementarity with respect to the players in \(S\).

In fact, following Bultow et al. [3], players \(i, j\) are said to be strategic complements whenever \(\partial_i^2 v(S, N, v) > 0\), for all \(S \subseteq N \setminus \{i, j\}\). They are said to be strategic substitutes whenever \(\partial_i^2 v(S, N, v) < 0\), for all \(S \subseteq N \setminus \{i, j\}\).
In this framework we define the average complementarity of players $i,j \in N$ as the following average of second-order differences:

$$\psi_{ij}(N,v) := \sum_{S \in \mathcal{N}(i,j)} \frac{n!}{(n-s)!} \frac{n!}{n!} \phi_{ij}^2(S, N, v), \text{ for all } i \neq j \in N. \quad (14)$$

Next proposition is a particular case of Proposition 4 in Flores et al. [10], and shows that the marginal contribution of a new player $i$ to the incumbent group $C$ is the sum of the a priori value $\phi_i(N,C,v_{TNC})$ in the social network of player $i$ which does not emerge from her relation with players in $C$, and the average complementarity between $i$ and $C$. Here $v_{TNC}$ stands for the restriction of the characteristic function $v$ to the set of players $N\setminus C$.

**Proposition 5.** Let $(N,\Gamma)$ be a social network, and $(N,v)$ a TU game that reflects its functionality. Then for every group $C \subseteq N$ and every player $i \in C$, the marginal contribution of player $i \in N \setminus C$ to the Myerson value of group $C$ equals:

$$\partial_i \psi_i^C(N, v, \Gamma) = \phi_i(N,C,v_{TNC}) + \psi_{iNC}(C,v_{TNC}), \text{ for all } i \in C. \quad (15)$$

7. Empirical methodology: an application

We will show now the potential of the Myerson group value in the analysis of a social network within a consulting company which has been considered by Borgatti [1] to illustrate the centrality group measures he proposed. We will describe how to gain more information about the elements that constitute the centrality of a given group. To be specific, on the one hand, by means of the theoretical results of Section 5 we will measure how much of the centrality of the groups in the top positions is due to their communication abilities and how much is due to their intermediation abilities. On the other hand, making use of the results of Section 6, and given a fixed incumbent group and a potential partner, we will differentiate between the centrality that the partner could aport to the group by himself and his contribution in terms of complementarity with the incumbent group.

Borgatti’s example consists of advice-seeking ties among members of a global company. The data were collected on a one to five strength-of-tie-scale, but for his analysis the author examined only the strongest ties (rated 5). The derived social network is then modeled by a graph, shown in Fig. 1 below.

Note that the information about the relations between the members of the organization should be obtained in some manner.

**Fig. 1.** Strong advice-seeking ties in a global consulting company [1].

Cross et al. [7] claim that a good way to make this possible is by using surveys:

“While social network information can be obtained in a variety of ways, the most pragmatic means in organizational settings is typically through surveys. Very informative social network diagrams can be generated from 10 to 15 min surveys assessing information or knowledge flow amongst members of a group. [...] Conducting a social network survey is a straightforward process of obtaining a list of all people in the defined network and simply asking all members of the group to characterize their relationship with each other.”

A guide to perform such surveys can be found in the appendix of the quoted paper.

The second step of the analysis consists in defining a cooperative game which adequately takes account the functionality of the network, which is previously known. In this sense, for example, Gomez et al. [12] define three games for modeling three scenarios, that in turn correspond to three appealing purposes: the overhead game, to mount a coordinated action; the messages game, to send information through the network via bilateral transmissions; and the conferences game, to send information through the network via multilateral transmissions.

More concretely, recall the motivational examples of the first section, referred now to Borgatti’s network. In the first one, a social network of informal relationships among the members of an organization has a critical influence on the formation of valuable working teams; then the conferences game, which reflects the level of communication among groups of two or more individual in a coalition, may be a good candidate to describe the functionality of the network. We propose below slight variations of the conferences game in order to consider the optimal size of a group in the context of advice seeking. In the second example, the online virtual network is used to share information and knowledge, and the goal of the organization is to spread their own information or innovation. Then, if we assume that communications are bilateral, the messages game gives a good model. In the last example, in which the goal of the organization is to pass a proposal and the social network captures the personal affinities between the subsidiaries’ boards of directors, the worth of each group can be described by means of a simple game $(N,u)$ that is in general non-symmetric. Frequently, the voting procedure can be described by means of a weighted simple game $(N,w)$, characterized by a vector of agents’ weights $w \in \mathbb{R}^n$ and a quota $q \in \mathbb{R}$ such that a coalition $S \subseteq N$ is winning when the sum of the weights of the players in $S$ meets or exceeds the quota that is requested to pass the proposal.

As commented, the social network of informal relationships between the members of an organization have “become a pervasive feature of organizations” (see [6]). Thus, the identification of a small group $C$ of agents who are able to lead the formation of efficient working teams, or whose deletion would disrupt most the ability of the social structure to form them, is crucial for the top managers of the organization. Hence, the last step of the analysis is the computation of the Myerson group value in order to identify those influential groups. This will be our goal in the rest of this section. Note in particular that if $(N,v)$ is a TU-game that describes the functionality of the network, each specific choice of game $(N,v)$ determines a centrality measure (the corresponding group value), and the final centrality figures and relative ordering depend on this specific choice.

If the purpose of the organization is the identification of a small group $C$ who is capable to integrate all the agents in the development of a common project, the overhead game describes appropriately the functionality of the social network. If the social network is intended to collaborate learning, the model game must take into account what is

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6. In the conferences game $v^{conf}(S) = 2^{|S|} - 1$ is number of subsets or conferences in $S$ with a cardinality greater than one, which reflects the level of communication among groups of two or more individual in a coalition $S$ for every $SCN$.

7. See Example 1 for a definition.
the ideal group size for that purpose. The main idea in the literature about this facet is “to keep groups midsized: small groups of 3 or less lack enough diversity and may not allow divergent thinking to occur. Groups that are too large create “freeloading” where not all members participate. A moderate size group of 4–5 is ideal”. We propose a trimmed version of the conferences game in which the worth of coalitions counts the subsets of midsized (4 or 5) conferences. Formally, the trimmed conferences game \((N, \nu^\text{conf−mid})\) is defined as

\[
\nu^\text{conf−mid}(S) = \min_{S \subseteq N \text{ with } |S| \geq 4, \nu^\text{conf−mid}(S)} \sum_{k=4}^{\min(|S|, 5)} \binom{|S|}{k},
\]

= 0, otherwise.

We analyze the most valuable groups up to size 4 when the purpose of the organization is to promote small collaborative learning groups, and we compare our results with those ones obtained by Borgatti in his analysis of the network of advice seeking. As a result we obtain that the trimmed conferences game gives results that are related to the Key Player Problem Negative analysis (KPP-Neg in the sequel).

Clearly, this social network has three distinguished subgroups that have not too many links connecting them: the first one, \(S_1\) made up by agents 1 to 11, the central one \(S_2\) made up by agents 12 to 22, and the third one \(S_3\) that comprises the remaining agents 23 to 32. Agents 7 in \(S_1\), 12 and 18 in \(S_2\) and 23 in \(S_3\) represent the unique connections among two consecutive groups. Note also that agents 7, 12, 18 and 23, together with player 19 in \(S_2\), form a chain that pass through the three subgroups. Considering the three subgroups in an isolated way, the two most central agents of \(S_1\) are 9 and 5; agents 19 and 13 are the two most central of \(S_2\), and agents 25 and 23 are the two most central of \(S_3\). All of them, according to both game theoretic centralities, with the overhead and the trimmed-conferences games. Apart from player 23, who is highly central inside his own subsociety, the remaining connection agents are more peripheral: agents 7 and 12 are in the fourth position inside their own subsocieties, and player 18 occupies the seventh position.

In a natural way, these nine agents (7, 9, 12, 13, 18, 19, 23, 25) are always in the top positions for all the considered criteria. When the underlying game is the trimmed-conferences game, the five best groups up to size 4 – arranged in decreasing order of importance – are listed in Table 1. We have followed Castro et al. [4] for estimating in polynomial time, via Monte Carlo simulation, the Myerson group value of the groups. Non-statistically significant differences (at a 99% of confidence level) between two groups centralities are labeled with an asterisk.

In this case, from an individual point of view, the four most central agents are 12, 7, 18 and 23, which are respectively the unique agents connecting consecutive subgroups.

With respect to groups of two agents, the most central is \((12, 18)\), which selects the two agents from \(S_2\) who connect this central subsociety with the other two extreme subgroups. Also remark that the two groups \((12, 23)\) and \((7, 18)\) that are tied in the second place select, in one hand, one of the agents in \(S_2\) who connects with one of the other two subgroups, and in the other, the player in the remaining subsociety that serves as a bridge. For a KPP-Neg analysis the algorithm proposed by Borgatti [1] selects the group \((7, 23)\), which is similar to \((12, 18)\) but selecting the intermediaries in \(S_2\) and \(S_3\). Note that the most central group of two agents – 12 and 18 – consists on the first and the third more central agents from and individual point of view, rather than the first and the second ones, 12 and 7. In this case, player 18 is a better partner for 12 than it is 7; note that the group \((12, 7)\) is in the sixth position. Although agents 12 and 7 are slightly more complements than 12 and 18, the centrality of agents 7 and 12 overlap a lot. Agent 7 without agent 12 limits himself to subsociety \(S_1\), and reduces

\[
\text{dramatically} \text{ his centrality in a } 99.71\% \text{ (from 32, 265.45 to 67.76), whereas agent 18 retains more centrality, although it is reduced in a } 89.57\% . \text{ Table 2 shows the decomposition of the marginal contribution of the top agents to the most central incumbent groups of one agent.}
\]

With respect to the most central groups of three and four agents, the groups with highest centrality consist on the three and four individually central agents: \((7, 12, 18)\) and \((7, 12, 18, 23)\). We should remark that groups \((7, 18, 19)\), \((7, 12, 19)\), \((12, 18, 19)\) and \((12, 19, 23)\) tie (at a 99% of confidence level) with the fifth group of three agents \((7, 19, 23)\). If we perform a decomposition similar to than of Table 2 we obtain results which are similar to those obtained for agent 7’s contribution to agent 12. That is, the power of the potential partner overlaps a lot with the incumbent group’s power, but they are very good complements. The percentages of \(\phi(N,C_\nu^\text{conf−mid})\) move from 1.17% to 1.90%.

For a KPP-Neg analysis the algorithm proposed selects \((7, 19, 23)\), in which the intermediary player 12 in \(S_2\) is replaced by 19 in \(S_2\) who is the most central player inside his subsociety. This group \((7, 19, 23)\) is in the fifth position according to the trimmed-conferences game theoretic group centrality.

Now, we will use the decomposition results obtained in Section 5 to decompose the centrality of agents \((7, 9, 12, 18, 19, 23, 25)\). The derived results are listed in Table 3, which also shows the decomposition for the two best groups of 2, 3 and 4 agents.

From an individual point of view, there are clearly two types of agents: \((7, 12, 18, 23)\) and \((9, 19, 25)\). The first group is characterized by a high percentage of betweenness power (from 80.44% to 84.04%), whereas the communication and betweenness power of the agents in the second group is much more balanced. This difference is due to the fact that agents 9, 19 and 25 occupy a central position in their respective subgroups when these are considered in an isolated way; whereas agents 7, 12, 18 and 23 represent the unique connections among two consecutive groups, but they occupy peripheral positions in their own subgroups. Note that although group \((12, 18)\) has more power than it has group \((12, 23)\) from a global point of view, agent 23 (which belong to subsociety \(S_3\)) increases agent 12’s communication power more than agent 18 (who belongs to his same subsociety) does. With respect to the relation between communication and betweenness power, note that communication power gains relevance as the size of the group increases.

### 8. Conclusions

Our interest has been centered in the assessments of groups in a social network within an organization. We have undertaken this problem by means of a twofold approach, whose first ingredient is the game-theoretic centrality measure defined by Gomez et al. [12]. Assuming 1) a social network where the agents are connected by a graph, and 2) that the purpose of the network is modeled by means of a cooperative game where the agents act as players; these authors defined an individual centrality measure by means of the (classical) Myerson value of the graph-restricted game.

The goal is the assessment of groups in a network organization by generalizing of this individual measure to a group valuation concept.
For that, we need our second basic ingredient, which is the Shapley game theoretic group centrality decompositions for the advice-seeking data based on \( (N, v^{coop−mst}) \). At this point it is worth to mention that our definition of the Myerson group value, which is our key concept for the assessment of groups.

After describing some motivating examples, we have investigated different aspects of the Myerson group value, as its possible usefulness to distinguish between two kinds of redundancy of agents – with respect to adjacency and to distance – or the combination of interaction and independence of an agent with respect to a given incumbent group in the search of its best partner. We end up our analysis by illustrating how to use empirically the methodology we propose for the identification of influential groups in a well-known advice-seeking network.

It is remarkable that one of the main features of our proposal is that it offers a wide framework to model many kinds of centrality measures. At this point it is worth to mention that our definition of group centrality is very flexible, in the sense that it can be based on any game capturing the functionality of the network. Other approaches can be obtained using other games and/or measures different from the Shapley value.

In forthcoming work we plan to address issues that are relevant to the practical use of our group valuation, as well as to explore interesting extensions. The problem of computing the Myerson group value of a given group \( C \) for big networks is interesting in itself. The computations for large networks could be undertaken using Castro-Gómez-Tejada algorithm [4], but more sophisticated methods will be needed when the number of nodes is bigger and/or the characteristic function of the game cannot be computed in polynomial time. Finally, another interesting extension of this work may include social networks with other type of relations among the agents.

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## Appendix A

### Proof of Proposition 1

It follows straightforward from the Shapley value self-duality \( \phi_i(N, v) = \phi_i(N, v^T) \), for every \( i \in N \), and for every \( n \)-person TU game \( (N, v) \) that \( \langle v^T \rangle = \langle v \rangle \). Then, \( \phi(N, v^T, g) = \phi(N, v, g) = \phi(N, v^T, g) = \phi(N, v, g) = \gamma(N, v, D) = \gamma(N, v, D) \).

### Proof of Lemma 1

Given a fixed player set \( N \) and a fixed social network \( (N, g) \) it is evident that the function \( \varphi_{n \rightarrow N} \) that associates to every TU-game \( (N, v) \) its \( g \)-graph-restriction \( (N, v_r) \) is a linear function. Hence, we have \( v_r = \sum_{S \subseteq N} \Delta^N(v, S)u_r^S \), where \( (N, u_r^S) \) is the \( g \)-graph-restriction of the unanimity game \( (N, u^S) \), which is given by (see Lemma 2.1 in Gomez et al. [12] for a proof)

\[
\hat{u}^S = 1 - \prod_{i=1}^r (1 - u^S_i).
\]

Here \( \hat{u}^S \) is the game defined by \( 1(S) = 1 \), for all \( \emptyset \neq S \subseteq N \), and \( \mathcal{M}_r(S) = \{S_1, \ldots, S_r\} \neq \emptyset \) is the set of minimal connection sets of \( S \) in \( g \). Note that this expression equals \( u^S_r \) when \( \mathcal{M}_r(S) \) has only one set.\(^9\) \( H(S) = S \) for the special case in which \( S \) is connected in \( g \).

\(^9\) If there are no cycles in \( g \) connecting two agents in \( S \), then there is only one smallest connected set \( H(S) \subseteq N \) which contains \( S \).
Now, given a fixed set of agents $C \subseteq N$ acting as a group, the transformation of games on the fixed player set $N$ into quotient games with respect to the partition $\pi_C = (C, \{j \in C\})$ is a linear function, and the quotient game derived from the unanimity game with respect to coalition $S \subseteq N$, $(N,u^S)$, is given by the unanimity game $(N_C,u^{NC})$. See Derks and Tijs \cite{8} for a proof. Then, if $(N, \Gamma)$ is a tree and $C \subseteq N$,

$$v_{\Gamma,C} = \sum_{S \subseteq \Gamma} \Delta^N(v, S)u^C_{NC} = \sum_{S \subseteq \Gamma} \Delta^N(v, S)u^{NC}/H_C(S). \quad (17)$$

Since the Shapley value is also linear, and $\phi_i(N,u^C) = \frac{1}{\Gamma}$, for all $i \in S$, $\phi_i(N,u^C) = 0$, for all $i \notin S$. Therefore, taking into account that the set of all coalitions $S$ such that $c \in N_C/H_C(S)$ equals the set of coalitions $S$ with $H_C(S) \cap C \neq \emptyset$, we have

$$\phi_C(N,v_C) := \phi_C(N_C,v_{\Gamma,C}) = \sum_{H_C(S) \cap C \neq \emptyset} \Delta^N(v, S)\left|N_C/H_C(S)\right| \quad \square$$

**Proof of Lemma 2.** The proof follows the same argument than that of the previous lemma. That is, $\phi_C(N,v_C) := \phi_C(N_C,v_{\Gamma,C}) = \sum_{S \subseteq \Gamma} \Delta^N(v, S)\phi_C(N_C,u^C_{NC})$. Now, we have to consider that $u^C_{NC} = 1 - \prod_{S \notin C} (1-u_{NC}^S)$, for every coalition $S \subseteq N$, and therefore $\phi_C(N_C,u^C_{NC}) = 0$, for every $S$ with $C \cap S = \emptyset$, and for every $S \in \mathcal{M}_R(S)$.

The reader interested in a explicit formula for $\phi_C(N_C,u^C_{NC})$ is referred to Gomez et al. \cite{12} (formula (11) in page 36); there, using expression (16) above, $\phi_i(N,u^C)$ is evaluated in the case in which there are alternative $v$ connecting the agents of $S$. \quad \square

**Proof of Proposition 4.** It follows straightforward from (7). \quad \square

References


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